3. Let V be a vector subspace of $\mathbf{R}^N$. Show that $T_x(V) = V$ if $x \in V$.

*4. Suppose that $f: X \longrightarrow Y$ is a diffeomorphism, and prove that at each x its derivative df_x is an isomorphism of tangent spaces.

5. Prove that $\mathbf{R}^k$ and $\mathbf{R}^l$ are not diffeomorphic if $k \neq l$.

6. The tangent space to S^1 at a point (a, b) is a one-dimensional subspace of $\mathbf{R}^2$. Explicitly calculate the subspace in terms of a and b. [The answer is obviously the space spanned by $(-b, a)$, but prove it.]

7. Similarly exhibit a basis for $T_p(S^2)$ at an arbitrary point $p = (a, b, c)$.

8. What is the tangent space to the paraboloid defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a}, 0, 0)$, where $(a > 0)$?

*9. (a) Show that for any manifolds X and Y,

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y).$$

(b) Let $f: X \times Y \longrightarrow X$ be the projection map $(x, y) \longrightarrow x$. Show that

$$df_{(x,y)} : T_x(X) \times T_y(Y) \longrightarrow T_x(X)$$

is the analogous projection $(v, w) \longrightarrow v$.

(c) Fixing any $y \in Y$ gives an injection mapping $f: X \longrightarrow X \times Y$ by $f(x) = (x, y)$. Show that $df_x(v) = (v, 0)$.

(d) Let $f: X \longrightarrow X'$, $g: Y \longrightarrow Y'$ be any smooth maps. Prove that

$$d(f \times g)_{(x,y)} = df_x \times dg_y.$$

*10. (a) Let $f: X \longrightarrow X \times X$ be the mapping $f(x) = (x, x)$. Check that $df_x(v) = (v, v)$.

(b) If Δ is the diagonal of $X \times X$, show that its tangent space $T_{(x,x)}(\Delta)$ is the diagonal of $T_x(X) \times T_x(X)$.

*11. (a) Suppose that $f: X \longrightarrow Y$ is a smooth map, and let $F: X \longrightarrow X \times Y$ be $F(x) = (x, f(x))$. Show that

$$dF_x(v) = (v, df_x(v)).$$

(b) Prove that the tangent space to graph (f) at the point $(x, f(x))$ is the graph of $df_x: T_x(X) \longrightarrow T_{f(x)}(Y)$.

*12. A *curve* in a manifold X is a smooth map $t \longrightarrow c(t)$ of an interval of $\mathbf{R}^1$ into X. The *velocity vector* of the curve c at time t_0—denoted simply

$dc/dt(t_0)$—is defined to be the vector $dc_{t_0}(1) \in T_{x_0}(X)$, where $x_0 = c(t_0)$ and $dc_{t_0}: \mathbf{R}^1 \longrightarrow T_{x_0}(X)$. In case $X = \mathbf{R}^k$ and $c(t) = (c_1(t), \ldots, c_k(t))$ in coordinates, check that

$$\frac{dc}{dt}(t_0) = (c_1'(t_0), \ldots, c_k'(t_0)).$$

Prove that every vector in $T_x(X)$ is the velocity vector of some curve in X, and conversely. [HINT: It's easy if $X = \mathbf{R}^k$. Now parametrize.]

§3 The Inverse Function Theorem and Immersions

Before we really begin to discuss the topology of manifolds, we must study the local behavior of smooth maps. Perhaps the best reason for always working with smooth maps (rather than continuous maps, as in non-differential topology) is that local behavior is often entirely specified, up to diffeomorphism, by the derivative. The elucidation of this remark is the primary objective of the first chapter.

If X and Y are smooth manifolds of the same dimension, then the simplest behavior a smooth map $f: X \longrightarrow Y$ can possibly exhibit around a point x is to carry a neighborhood of x diffeomorphically onto a neighborhood of $y = f(x)$. In such an instance, we call f a *local diffeomorphism* at x. A necessary condition for f to be a local diffeomorphism at x is that its derivative mapping $df_x: T_x(X) \longrightarrow T_y(Y)$ be an isomorphism. (See Exercise 4 in Section 2). The fact that this linear condition is also sufficient is the key to understanding the remark above.

The Inverse Function Theorem. Suppose that $f: X \longrightarrow Y$ is a smooth map whose derivative df_x at the point x is an isomorphism. Then f is a local diffeomorphism at x.

The Inverse Function Theorem is a truly remarkable and valuable fact. The derivative df_x is simply a single linear map, which we may represent by a matrix of numbers. This linear map is nonsingular precisely when the determinant of its matrix is nonzero. Thus the Inverse Function Theorem tells us that the seemingly quite subtle question of whether f maps a neighborhood of x diffeomorphically onto a neighborhood of y reduces to a trivial matter of checking if a single number—the determinant of df_x—is nonzero!

You have probably seen a proof of the Inverse Function Theorem for the special case when X and Y are open subsets of Euclidean space. One may be found in any text on calculus of several variables—for example, Spivak [2]. You should easily be able to translate the Euclidean result to the manifold setting by using local parametrizations.